4.6 Quadrature Amplitude Modulation (QAM)

4.70. We are now going to define a quantity called the "bandwidth" of a signal. Unfortunately, in practice, there isn't just one definition of bandwidth.

Definition 4.71. The **bandwidth** (BW) of a signal is usually calculated from the differences between two frequencies (called the bandwidth limits). Let's consider the following definitions of bandwidth for real-valued signals [3, p 173]

$f_{max} - f_{min}$ f_{max} f_{min}

(a) **Absolute bandwidth**: Use the highest frequency and the lowest frequency in the positive-f part of the signal's nonzero magnitude spectrum.

- This uses the frequency range where 100% of the energy is confined.
- We can speak of absolute bandwidth if we have ideal filters and unlimited time signals.

(b) **3-dB bandwidth** (**half-power bandwidth**): Use the frequencies where the signal power starts to decrease by 3 dB (1/2).

- The magnitude is reduced by a factor of $1/\sqrt{2}$.

(c) **Null-to-null bandwidth**: Use the signal spectrum's first set of zero crossings.

(d) **Occupied bandwidth**: Consider the frequency range in which $X\%$ (for example, 99%) of the energy is contained in the signal's bandwidth.

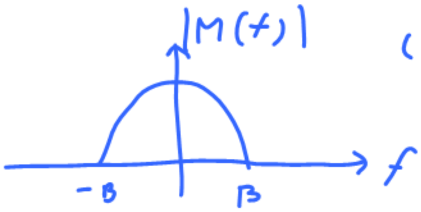
(e) **Relative power spectrum bandwidth**: the level of power outside the bandwidth limits is reduced to some value relative to its maximum level.

- Usually specified in negative decibels (dB).
- For example, consider a 200-kHz-BW broadcast signal with a maximum carrier power of 1000 watts and relative power spectrum bandwidth of -40 dB (i.e., 1/10,000). We would expect the station's power emission to not exceed 0.1 W outside of $f_c \pm 100$ kHz.

$x(t) = m(t)\cos(2\pi f_c t)$

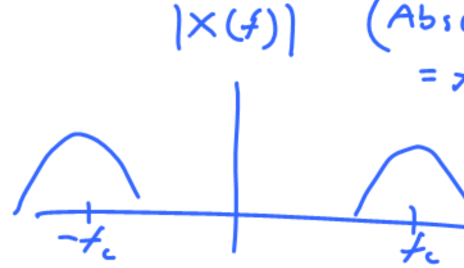
DSB-SC

Example 4.72. Message bandwidth and the transmitted signal bandwidth



(Absolute) BW = $B - 0$

$= B$



(Absolute) BW

$= f_{max} - f_{min}$

$= 2B$

4.73. Rough Approximation: If $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2 Hz, respectively, the bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz.

This result follows from the application of the width property[18] of convolution[19] to the convolution-in-frequency property.

Consequently, if the bandwidth of $g(t)$ is B Hz, then the bandwidth of $g^2(t)$ is $2B$ Hz, and the bandwidth of $g^n(t)$ is nB Hz. We mentioned this property in 2.42.

4.74. BW Inefficiency in DSB-SC system: Recall that for real-valued baseband signal $m(t)$, the conjugate symmetry property from 2.30 says that

$$M(-f) = (M(f))^*.$$

The DSB spectrum has two sidebands: the upper sideband (USB) and the lower sideband (LSB), both containing complete information about the baseband signal $m(t)$. As a result, DSB signals occupy twice the bandwidth required for the baseband.

4.75. To improve the spectral efficiency of amplitude modulation, there exist two basic schemes to either utilize or remove the spectral redundancy:

(a) Single-sideband (SSB) modulation, which removes either the LSB or the USB so that for one message signal $m(t)$, there is only a bandwidth of B Hz.

(b) Quadrature amplitude modulation (QAM), which utilizes spectral redundancy by sending two messages over the same bandwidth of $2B$ Hz.

We will only discussed QAM here. SSB discussion can be found in [3, Sec 4.4], [13, Section 3.1.3] and [4, Section 4.5].

[18] This property states that the width of $x * y$ is the sum of the widths of x and y.

[19] The width property of convolution does not hold in some pathological cases. See [4, p 98].

Definition 4.76. In ***quadrature amplitude modulation*** (***QAM***) or ***quadrature multiplexing***, two baseband real-valued signals $m_1(t)$ and $m_2(t)$ are transmitted simultaneously via the corresponding QAM signal:

Form 1:

$$x_{\text{QAM}}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t).$$

I-channel

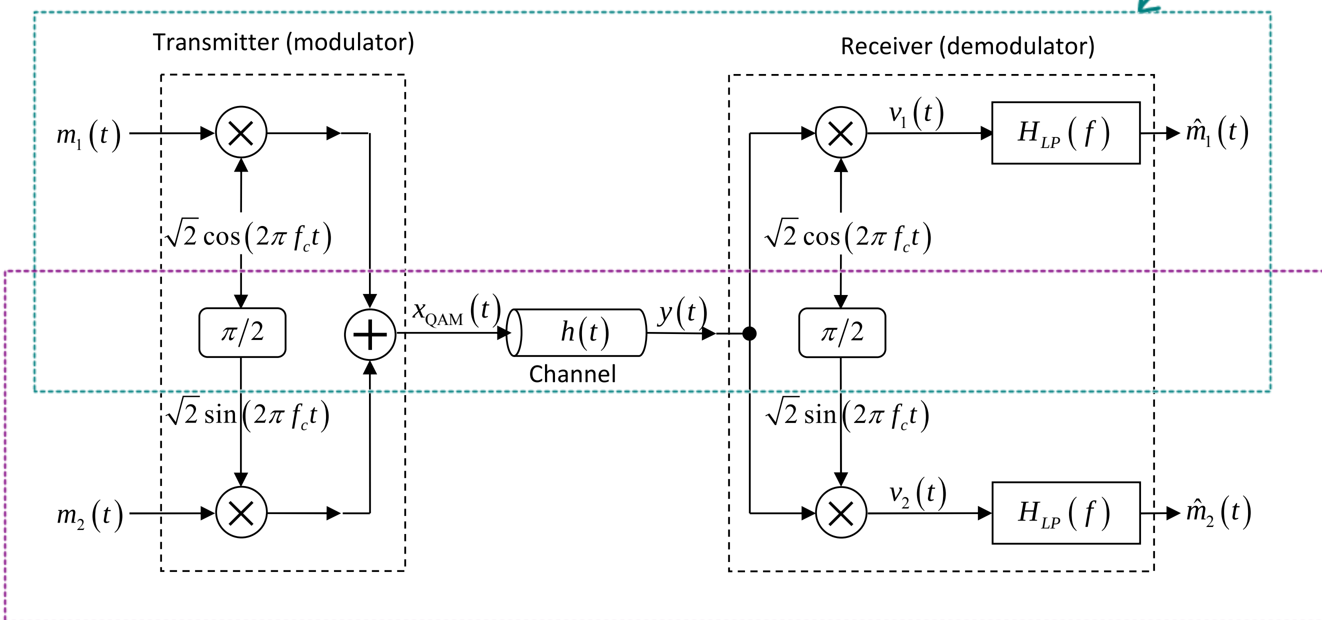


Figure 26: QAM Scheme

Q-channel

- QAM operates by transmitting two DSB signals via carriers of the same frequency but in phase quadrature.
- Both modulated signals simultaneously occupy the same frequency band.
- The "cos" (upper) channel is also known as the ***in-phase*** (***I***) channel and the "sin" (lower) channel is the ***quadrature*** (***Q***) channel.

4.77. ***Demodulation***: Under the usual assumption ($B < f_c$), the two baseband signals can be separated at the receiver by synchronous detection:

$$\text{LPF}\left\{x_{\text{QAM}}(t)\sqrt{2}\cos(2\pi f_c t)\right\} = m_1(t) \tag{54}$$

$$\text{LPF}\left\{x_{\text{QAM}}(t)\sqrt{2}\sin(2\pi f_c t)\right\} = m_2(t) \quad \leftarrow \text{[HW 7]} \tag{55}$$

To see (54), note that

$$
\begin{aligned}
v_1(t) &= x_{\text{QAM}}(t)\sqrt{2}\cos(2\pi f_c t) \\
&= \left(m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)\right)\sqrt{2}\cos(2\pi f_c t) \\
&= m_1(t)\,2\cos^2(2\pi f_c t) + m_2(t)\,2\sin(2\pi f_c t)\cos(2\pi f_c t) \\
&= m_1(t)\left(1+\cos(2\pi(2f_c)t)\right) + m_2(t)\sin(2\pi(2f_c)t) \\
&= m_1(t) + m_1(t)\cos(2\pi(2f_c)t) + m_2(t)\cos(2\pi(2f_c)t - 90^\circ)
\end{aligned}
$$

$LPF\{v_1(t)\} = m_1(t) + 0 + 0$

- Observe that $m_1(t)$ and $m_2(t)$ can be separately demodulated.

Example 4.78. $3\sqrt{2}\cos(2\pi f_c t) + 4\sqrt{2}\sin(2\pi f_c t)$

Phasor representation

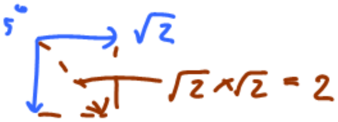
calculator

$\Leftrightarrow 3\sqrt{2}\angle 0^\circ + 4\sqrt{2}\angle -90^\circ = 5\sqrt{2}\angle -53^\circ$

$\Leftrightarrow 5\sqrt{2}\cos(2\pi f_c t - 53^\circ)$

Example 4.79. $(1)\sqrt{2}\cos(2\pi f_c t) + (1)\sqrt{2}\sin(2\pi f_c t)$

$\Leftrightarrow \sqrt{2}\angle 0^\circ + \sqrt{2}\angle(-90^\circ) = 2\angle -45^\circ$



$\Leftrightarrow 2\cos(2\pi f_c t - 45^\circ)$

4.80. $m_1\sqrt{2}\cos(2\pi f_c t) + m_2\sqrt{2}\sin(2\pi f_c t)$

$\Leftrightarrow m_1\sqrt{2}\angle 0^\circ + m_2\sqrt{2}\angle -90^\circ = \sqrt{2}(m_1 - jm_2) = \sqrt{2}E\angle\phi$

$\Leftrightarrow E\sqrt{2}\cos(2\pi f_c t + \phi)$

4.81. Sinusoidal form (envelope-and-phase description [3, p. 165]):

Form ※2

$$x_{\text{QAM}}(t) = \sqrt{2}E(t)\cos(2\pi f_c t + \phi(t)),$$

where

$$
\begin{aligned}
\text{envelope:}\quad & E(t) = |m_1(t) - jm_2(t)| = \sqrt{m_1^2(t) + m_2^2(t)} \\
\text{phase:}\quad & \phi(t) = \angle\left(m_1(t) - jm_2(t)\right)
\end{aligned}
$$

calculator

rectangular — polar

$m_1(t) - jm_2(t) = E(t)\angle\phi(t)$

Example 4.82. In a QAM system, the transmitted signal is of the form

$$x_{\text{QAM}}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t).$$

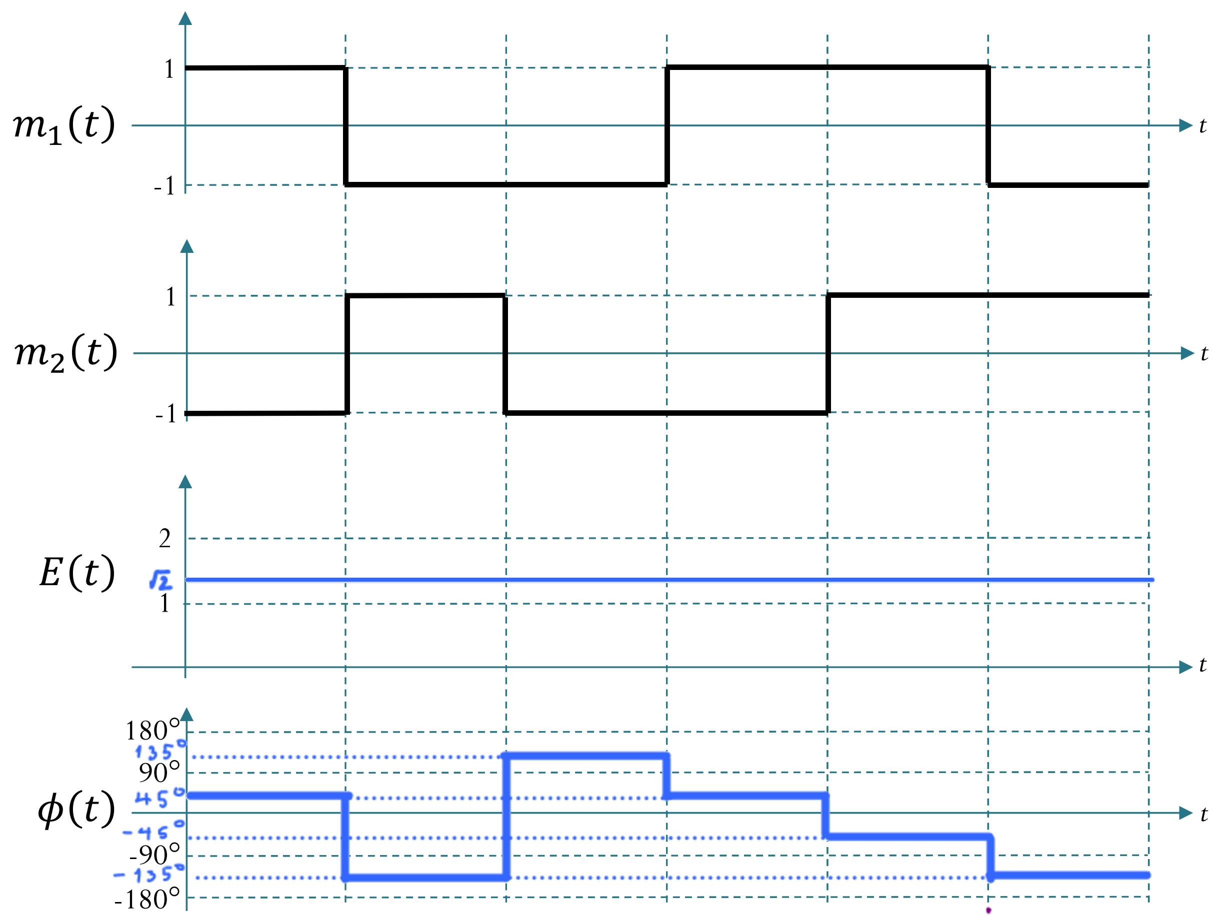
Here, we want to express $x_{\text{QAM}}(t)$ in the form

$$x_{\text{QAM}}(t) = \sqrt{2}E(t)\cos(2\pi f_c t + \phi(t)),$$

where $E(t) \geq 0$ and $\phi(t) \in (-180°, 180°]$.

m_1	m_2	$m_1 - jm_2$
1	1	$1 - j = \sqrt{2}\angle -45°$
1	-1	$1 + j = \sqrt{2}\angle 45°$
-1	1	$-1 - j = \sqrt{2}\angle -135°$
-1	-1	$-1 + j = \sqrt{2}\angle 135°$

Consider $m_1(t)$ and $m_2(t)$ plotted in the figure below. Draw the corresponding $E(t)$ and $\phi(t)$.



$e^{j\alpha} = \cos\alpha + j\sin\alpha$

$\cos(2\pi f_c t - 90°)$

$\cos\alpha = \text{Re}\{e^{j\alpha}\}$

4.83. $m_1\sqrt{2}\cos(2\pi f_c t) + m_2\sqrt{2}\sin(2\pi f_c t)$

$$= m_1\sqrt{2}\,\text{Re}\{e^{j2\pi f_c t}\} + m_2\sqrt{2}\,\text{Re}\{\underbrace{e^{j(2\pi f_c t - 90°)}}_{e^{j2\pi f_c t}\underbrace{e^{j(-90°)}}_{-j}}\}$$

$$= \sqrt{2}\,\text{Re}\{(m_1 - jm_2)e^{j2\pi f_c t}\}$$

4.84. Complex form:

Form 3:
$$x_{\text{QAM}}(t) = \sqrt{2}\text{Re}\left\{(m(t))\, e^{j2\pi f_c t}\right\}$$

where[20] $m(t) = m_1(t) - jm_2(t)$.

- We refer to $m(t)$ as the ***complex envelope*** (or ***complex baseband signal***) and the signals $m_1(t)$ and $m_2(t)$ are known as the ***in-phase*** and ***quadrature(-phase)*** components of $x_{\text{QAM}}(t)$.
- The term "quadrature component" refers to the fact that it is in phase quadrature ($\pi/2$ out of phase) with respect to the in-phase component.
- Key equation:

$$\text{LPF}\left\{\underbrace{\left(\text{Re}\left\{m(t) \times \sqrt{2}e^{j2\pi f_c t}\right\}\right)}_{x(t)} \times \left(\sqrt{2}e^{-j2\pi f_c t}\right)\right\} = m(t). \quad \text{[HW7e]}$$

4.85. Three equivalent ways of saying exactly the same thing:

(a) the complex-valued envelope $m(t)$ complex-modulates the complex carrier $e^{j2\pi f_c t}$,

- So, now you can understand what we mean when we say that a complex-valued signal is transmitted.

(b) the real-valued amplitude $E(t)$ and phase $\phi(t)$ real-modulate the amplitude and phase of the real carrier $\cos(2\pi f_c t)$,

(c) the in-phase signal $m_1(t)$ and quadrature signal $m_2(t)$ real-modulate the real in-phase carrier $\cos(2\pi f_c t)$ and the real quadrature carrier $\sin(2\pi f_c t)$.

---

[20]If we use $-\sin(2\pi f_c t)$ instead of $\sin(2\pi f_c t)$ for $m_2(t)$ to modulate,

$$\begin{aligned} x_{\text{QAM}}(t) &= m_1(t)\sqrt{2}\cos(2\pi f_c t) - m_2(t)\sqrt{2}\sin(2\pi f_c t) \\ &= \sqrt{2}\,\text{Re}\left\{m(t)\, e^{j2\pi f_c t}\right\} \end{aligned}$$

where

$$m(t) = m_1(t) + jm_2(t).$$

4.86. References: [3, p 164–166, 302–303], [13, Sect. 2.9.4], [4, Sect. 4.4], and [8, Sect. 1.4.1]

4.87. Question: In engineering and applied science, measured signals are real. Why should real measurable effects be represented by complex signals?

Answer: One complex signal (or channel) can carry information about two real signals (or two real channels), and the algebra and geometry of analyzing these two real signals as if they were one complex signal brings economies and insights that would not otherwise emerge. [8, p. 3]

4.7 Suppressed-Sideband Amplitude Modulation

4.88. The upper and lower sidebands of DSB are uniquely related by symmetry about the carrier frequency, so either one contains all the message information. Hence, transmission bandwidth can be cut in half if one sideband is suppressed along with the carrier.

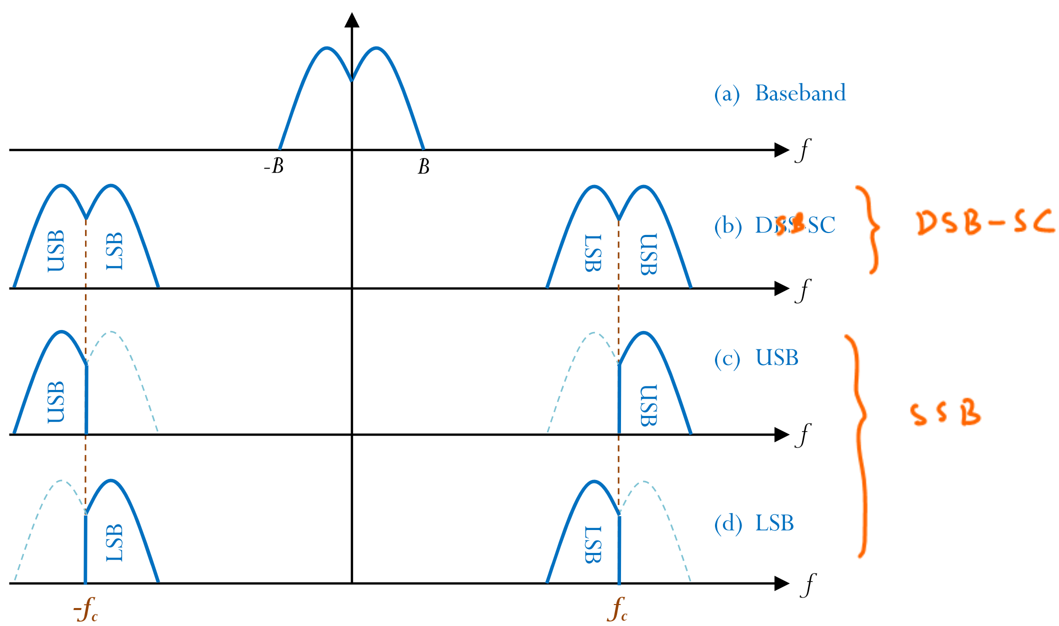


Figure 27: SSB spectra from suppressing one DSB sideband.

Definition 4.89. Conceptually, in **single-sideband modulation** (**SSB**), a sideband filter suppresses one sideband before transmission. [3, p 185–186]

(a) If the filter removes the lower sideband, the output spectrum consists of the upper sideband (USB) alone. Mathematically, the time domain representation of this SSB signal is

$$x_{\text{USB}}(t) = m(t)\sqrt{2}\cos(2\pi f_c t) - m_h(t)\sqrt{2}\sin(2\pi f_c t). \tag{56}$$

where $m_h(t)$ is the **Hilbert transform** of the message:

$$m_h(t) = \mathcal{H}\{m(t)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{m(\tau)}{t-\tau}d\tau = m(t) * \frac{1}{\pi t}. \tag{57}$$

(b) If the filter removes the upper sideband, the output spectrum consists of the lower sideband (LSB) alone. Mathematically, the time domain representation of this SSB signal is

$$x_{\text{LSB}}(t) = m(t)\sqrt{2}\cos(2\pi f_c t) + m_h(t)\sqrt{2}\sin(2\pi f_c t). \tag{58}$$

Derivation of the time-domain representation is given in 4.91.

4.90. There are a couple of important Fourier transform pairs[21] that haven't been discussed earlier.

(a) For the signum function,

$$\operatorname{sgn}(t) = \left\{\begin{array}{ll} 1, & t > 0 \\ -1, & t < 0 \end{array}\right\} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{j\pi f} \tag{59}$$

To remember this, simply note that $\frac{d}{dt}\operatorname{sgn}(t) = 2\delta(t)$. Therefore,

$$\mathcal{F}\left\{\frac{d}{dt}\operatorname{sgn}(t)\right\} = \mathcal{F}\{2\delta(t)\} \equiv 2. \tag{60}$$

From the time differentiation property, we also have

$$\mathcal{F}\left\{\frac{d}{dt}\operatorname{sgn}(t)\right\} = j2\pi f\mathcal{F}\{\operatorname{sgn}(t)\} \tag{61}$$

Equating (60) and (61), we get (59). Note that such method is deceptively simple but does not highlight the difficulties inherent in the functions involved.

(b) For the unit-step function, because $u(t) = \frac{1+\operatorname{sgn}(t)}{2}$, we have

$$u(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}.$$

[21] Derivation of these pairs are not straight-forward. For those who are interested, please see B.L. Burrows and D.J. Colwell (1990): The Fourier transform of the unit step function, International Journal of Mathematical Education in Science and Technology, 21:4, 629–635

(c) Applying the duality theorem to (59), we get

$$\frac{1}{\pi t} = h(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} H(f) = -j\,\mathrm{sgn}(f) = \begin{cases} -j, & f > 0 \\ j, & f < 0 \end{cases} = \begin{cases} 1 \cdot e^{-j\frac{\pi}{2}}, & f > 0 \\ 1 \cdot e^{j\frac{\pi}{2}}, & f < 0 \end{cases}$$

4.91. Let's define the right half and left half of $M(f)$ as $M_+(f)$ and $M_-(f)$, respectively. Observe that

$$M_+(f) \equiv \left\{ \begin{array}{ll} M(f), & f > 0 \\ 0, & f < 0 \end{array} \right\} = M(f)\,u(f) = M(f)\,\frac{1}{2}\,(1 + \mathrm{sgn}(f)) \tag{62}$$

From (57), applying the convolution-in-time property to the Hilbert transform of the message, we have

$$m(t) * \frac{1}{\pi t} = m_h(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M_h(f) = M(f) \times (-j\,\mathrm{sgn}(f)). \tag{63}$$

Replacing $M(f)\,\mathrm{sgn}(f)$ in (62) with $jM_h(f)$, we have

$$M_+(f) = \frac{1}{2}\,(M(f) + jM_h(f)).$$

Similarly,

$$M_-(f) \equiv \left\{ \begin{array}{ll} 0, & f > 0 \\ M(f), & f < 0 \end{array} \right\} = M(f)\,u(-f) = M(f)\,\frac{1}{2}\,(1 - \mathrm{sgn}(f)) = \frac{1}{2}\,(M(f) - jM_h(f)).$$

Now, by the frequency-domain construction (in Figure 27c),

$$\begin{aligned} X_{\mathrm{USB}}(f) &= AM_+(f - f_c) + AM_-(f - (-f_c)), \\ &= \frac{A}{2}\,(M(f - f_c) + jM_h(f - f_c)) + \frac{A}{2}\,(M(f + f_c) - jM_h(f + f_c)), \\ &= \frac{A}{2}\,(M(f - f_c) + M(f + f_c)) - \frac{A}{2j}\,(M_h(f - f_c) - M_h(f + f_c)), \end{aligned}$$

With $A = \sqrt{2}$, the inverse Fourier transform is (56).

4.92. An SSB signal can be synchronously (coherently) demodulated just like DSB-SC signals. For example, multiplication of a USB signal by $\sqrt{2}\cos(2\pi f_c t)$ shifts its spectrum to the left and right by f_c, creating $M(f)$ around $f = 0$. Low-pass filtering of this signal yields the desired baseband signal. The case is similar with LSB signals.

Mathematically,

$$\begin{aligned} x_{\mathrm{SSB}}(t)\,\sqrt{2}\cos(2\pi f_c t) &= \left(m(t)\sqrt{2}\cos(2\pi f_c t) \mp m_h(t)\sqrt{2}\sin(2\pi f_c t)\right)\sqrt{2}\cos(2\pi f_c t) \\ &= m(t)\,(1 + \cos(2\pi\,(2f_c)\,t)) \mp m_h(t)\sin(2\pi\,(2f_c)\,t) \\ &= m(t) + m(t)\cos(2\pi\,(2f_c)\,t) \mp m_h(t)\sin(2\pi\,(2f_c)\,t) \end{aligned}$$

Observe that

(a) If $m(t)$ is band-limited to B, then $m_h(t)$ is also band-limited to B because, from (63), we know that $M_h(f) = M(f) \times (-j\,\mathrm{sgn}(f))$. Therefore, the LPF that eliminates $m(t)\cos(2\pi(2f_c)t)$ will also eliminate $m_h(t)\sin(2\pi(2f_c)t)$.

(b) The product $x_{\mathrm{SSB}}(t)\sqrt{2}\cos(2\pi f_c t)$ yields the baseband signal and another SSB signal with twice the carrier frequency.

4.93. An ideal Hilbert transformer (Hilbert phase shifter) is unrealizable (or realizable only approximately). This is due to an abrupt phase change of π at zero frequency.

Practical approximation of this ideal phase shifter still works fine when the message $m(t)$ has a dc null and very little low-frequency content.

Definition 4.94. In **vestigial-sideband modulation** (**VSB**) (or asymmetric sideband [5]), one sideband is passed almost completely while just a trace, or vestige, of the other sideband is included. [3, p 191–192]

4.95. In (analog) television video transmission, an AM wave is applied to a vestigial sideband filter. This modulation scheme is called **VSB plus carrier** (**VSB + C**). [3, p 193]

- The unsuppressed carrier allows for envelope detection, as in AM
  - Distortionless envelope modulation actually requires symmetric sidebands, but VSB + C can deliver a fair approximation.